INVERSE REASONING WITH QUANTITATIVE UNKNOWNS

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IDR²eAM Project

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IDR²eAM Project

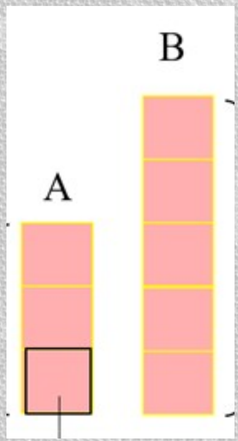
Investigating Differentiated Instruction and Relationships between Rational Number Knowledge and Algebraic Reasoning in Middle School

- **Purposes: To study…**
  - How to differentiate instruction for cognitively diverse middle school students
  - How students' rational number knowledge and algebraic reasoning are related
- **Phase I (2 yrs):** Three design experiments after school
  - 6-9 seventh & eighth grade students selected for cognitive diversity
  - 18 episodes each, 22 students total

Purpose of Talk

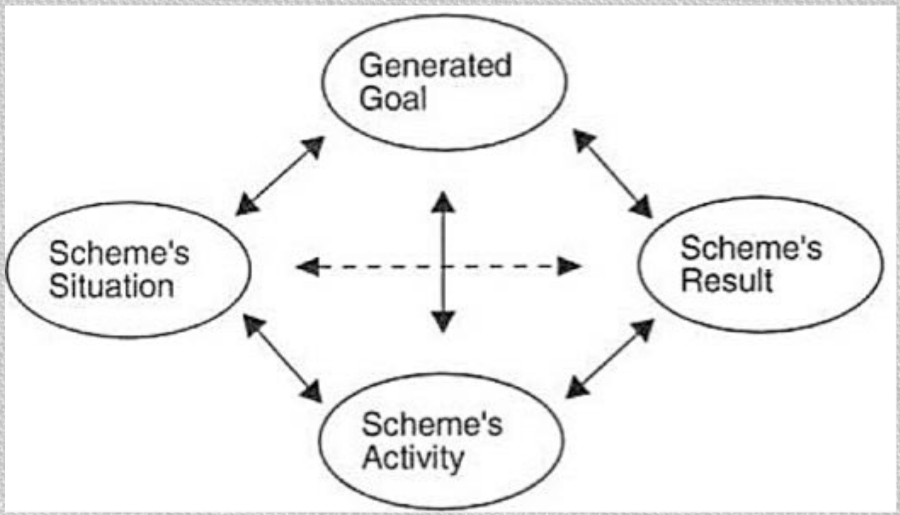
- How was reflective abstraction involved in constructing and stabilizing reciprocal reasoning with quantitative unknowns?
  - What about students who did not construct reciprocal reasoning?

- **Reciprocal reasoning:** seeing, justifying, and using the idea that if 3/5 of height B is height A, then B must be 5/3 of A (more soon)



Mathematical Learning

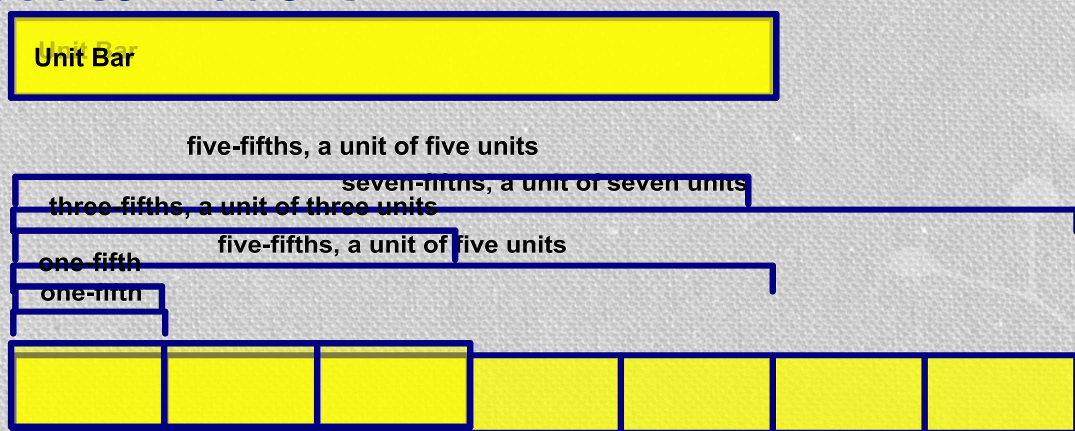
- I view learning in the context of **making accommodations in schemes in on-going interaction in one's experiential world**
- **Schemes:** goal-directed ways of operating that include a perceived situation, activity, and perceived result.
- **Accommodations:** reorganizations of and modifications in schemes



Steffe & Olive (2010, p. 23)

The Iterative Fraction Scheme

- Fractions are multiples of unit fractions
- Improper fractions are also whole numbers with additional fractional parts
- Students can think about and operate with fractions beyond 1 without conflations.



Reflective abstraction (Piaget, 1980)

- **Reflecting abstraction:** reorganization of mental operations and projection of them to new level
  - Motor behind construction of schemes (Thompson, 1994)
  - Motor behind some accommodations
  - Motor behind interiorization of results of schemes = formation of concepts
- **Reflected abstraction:** deliberate thematization of mental operations
  - "Looking back" on one's ways of thinking to discern patterns and structure

Phase I students: 9 out of 22 students with IFS

	IFS, initially	RR	IR
Experiment 1	3	3	
Experiment 2	3	2	1
Experiment 3	3	2	1

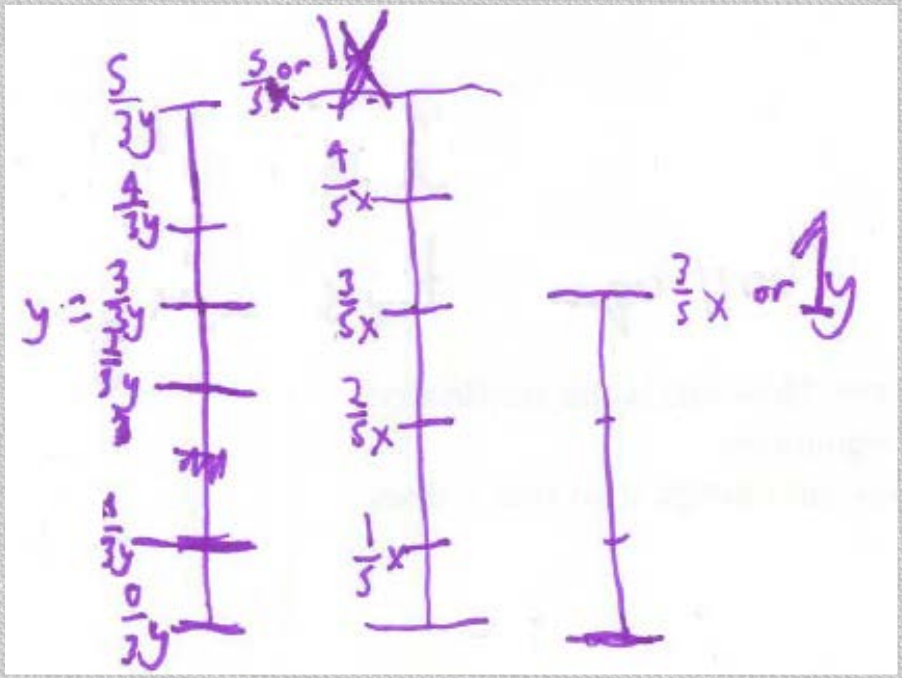
- **Conjecture:** These students will construct reciprocal reasoning with quantitative unknowns.
- **Finding:**
  - 7 of the 9 students constructed reciprocal reasoning;
  - 2 constructed inverse reasoning with fractional relationships between quantitative unknowns.

Reciprocal Reasoning with Quantitative Unknowns

Problem (summary): The unknown height of the sunflower is 3/5 the unknown height of the fern.

- Draw a picture.
- Write and explain equations.

- All wrote an equation with 3/5.
- Two students knew that 5/3 would be the other relationship to use; none could justify it originally.
- Seven students experienced an insight that each 1/5 of the fern ht was 1/3 of the sunflower ht.



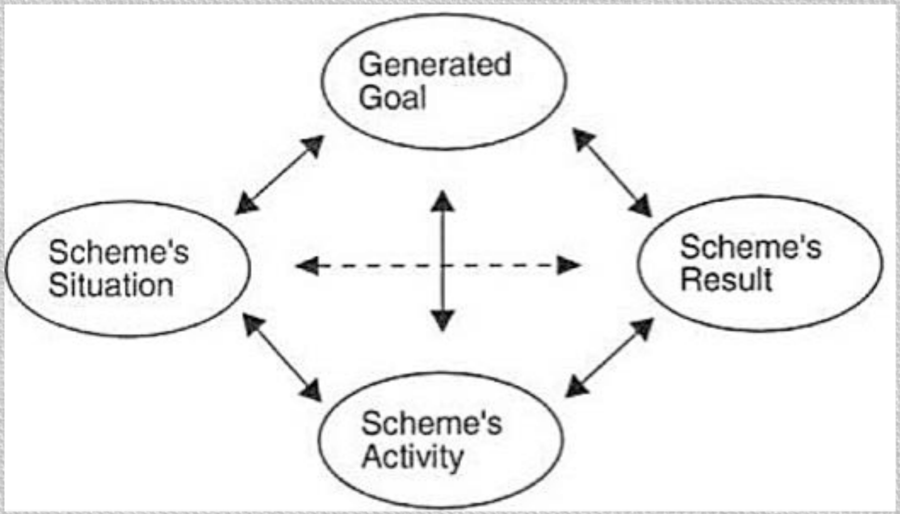
Let x = fern ht, y = sunflower ht

y = 3/5x AND x = 5/3y

Reciprocal Reasoning Scheme

Develop equations
for 2 Unknown Problems

Make fractions
of unknowns



Use IFS recursively

Create 2 equations
with reciprocal
relationship

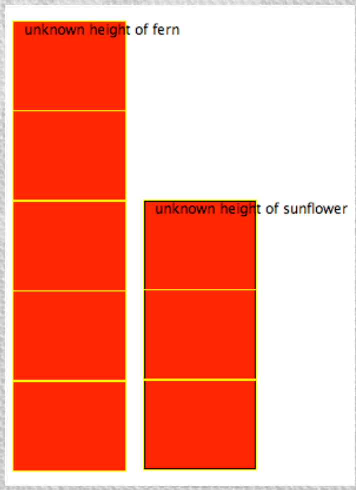
- 7 made initial construction
- 4 stabilized their schemes

Account in terms of Reflecting and Reflected Abstraction

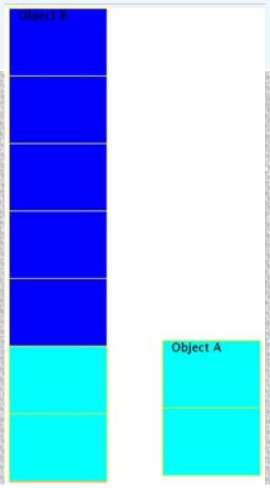
- **Initial construction of reciprocal reasoning scheme:** accommodation in iterative fraction scheme – province of REFLECTING ABSTRACTION
- **Subsequent stabilization of reciprocal reasoning scheme:** deliberate thematization – province of REFLECTED ABSTRACTION
- **But what happened with 2 of the 9 students, Amanda (e2) and Katrina (e3)?**

Inverse Reasoning with Quantitative Unknowns: Amanda (e2) and Katrina (e3)

	Equations for Fern Sunflower Heights Problem f = fern ht; s = sunflower ht
Amanda	f ÷ 5 • 3 = s, s ÷ 3 • 5 = f ←then rejected this Sequence: s + 2/5 = f, s + f2/5 = f, s + 2/3 = f, and s + 2/3s = f
Katrina	f ÷ 5 • 3 = s, s ÷ 3 • 5 = f

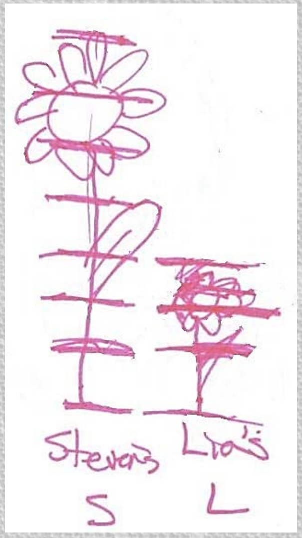


Katrina, ht A is 2/7 of ht B problem:
A + A + A + 1/2A = B



Katrina’s Follow-up Interview

- **Problem:** Steven and Lia are each growing a sunflower plant. The height of each of their plants is unknown, and the height of Lia’s plant measured in inches is 3/7 the height of Steven’s plant measured in inches. [Draw a picture, write equations.]
- Katrina’s initial equations:
  - S ÷ 7 x 3 = L
  - “an opposite of this,” L ÷ 3 x 7 = S
- With prompting to use a fraction:
  - 3/7S = L



Katrina's equations so far:

$S \div 7 \times 3 = L$

$L \div 3 \times 7 = S$

$3/7S = L$

Inverse v. Reciprocal Reasoning

- **Reciprocal (stable):** package of two relationships
- **Reciprocal (initial construction):** package of relationships that needs to be recreated
- **Inverse:** process represented to produce each height from the other, usually with whole number multiplication and division
  - From Inhelder and Piaget's inversion aspect of reversibility (1958)

Oops!

- **Conjecture:** Need to support construction/stabilization of iterative fraction scheme before working on reciprocal reasoning
- Too much attention to designing to support reflected abstraction v. designing to support reflecting abstraction
- Is it a short-term learning goal for students like Amanda and Katrina to stabilize their iterative fraction schemes?

THANK YOU!

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- And BIG thanks to others on the **IDR²eAM** project team: Fetiye Aydeniz, Rebecca Borowski, Mark Creager, Ayfer Eker, Sharon Hoffman, Robin Jones, Rob Matyska, Pai Suksak
- *What IDR²eAM stands for:*
  Investigating **D**ifferentiated Instruction and **R**elationships between **R**ational Number Knowl**e**dge and **A**lgebraic Reasoning in **M**iddle School
- http://www.indiana.edu/~idream/

Revised Learning Trajectory for RR with QU

Hypothetical addition to LT for MC3 students (insert this row at the start of LT)

Significant events	Description of the reasoning	Learning Processes	Instructional Supports (Examples)
Constructing or solidifying an iterative fraction scheme and reversible iterative fraction scheme	Students view any fraction as a multiple of a unit fraction, as well as a number of units of 1 and a proper fraction.	Initial construction: An accommodation in students' fraction schemes in which students see the result of their scheme as a multiplicative relationship, rather than as parts out of a whole. Stabilization: Reflected abstraction where students engage in repeated experience and retroactive thematization to examine and use improper fractions as usable numbers.	• Asking students to draw 7/5 of a bar, or to draw the whole bar given 7/5 of it • Asking students to iterate unit fractions in order to create improper fractions • Asking students to, e.g., use ninths to draw a bar a little bit longer than an 8/8-bar